Tractable s-widths in weighted Wiener spaces

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Function Spaces, Analysis and Approximation

06.02.2023

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Funded by

Europäische Union

Europa fördert Sachsen.
ESF
Europäischer Sozialfonds

Freistaat Sachsen

Diese Maßnahme wird mitfinanziert durch Steuermittel auf der Grundlage des vom Sächsischen Landtag beschlossenen Haushaltes.

Asymptotic characteristics

For $n \in \mathbb{N}_0$, and $X(\Omega), Y$ quasi-Banach function spaces with a continuous linear embedding $T : X \to Y$ the following (quasi) s-Numbers are defined:

- Sampling numbers (linear and non-linear)

$$\varrho_n(X)_Y = \inf_{t_1 \dots t_n \in \Omega} \inf_{R:\mathbb{C}^n \to Y} \sup_{\|f\|_X \le 1} \left\| f - R\big(f(t_1) \dots f(t_n)\big) \right\|_Y \tag{1}$$

- Gelfand numbers

$$c_n(X)_Y = \inf \left\{ \sup_{f \in B_X \cap M} \|f\|_Y : M \subset X \text{ linear subspace with } \operatorname{codim} M < n \right\} \tag{2}$$

- best trigonometric m-term approximation

$$\sigma_n(X)_Y := \sup_{\|f\|_X \le 1} \inf_{s \in \Sigma_n} \|f - s\|_Y \tag{3}$$

Main result - Besov spaces with dominating mixed smoothness

For Besov spaces with dominating mixed smoothness $S^r_{p,\theta}B(\mathbb{T}^d)$ we get

Theorem 1

Let $n, d \in \mathbb{N}$ *and* $(p,\theta) \in \{(p,\theta) :\ 2 \le p < \infty, 0 < \theta \le 1\} \setminus (2,1)$. *Then it holds*

$$\sigma_n(S^{1/\theta-1/2}_{p,\theta}B(\mathbb{T}^d))_\infty \le C\,(1/\theta - 1/2 - 1/p)^{-1/2} d\, n^{1/2-1/\theta} \log(dn)^{1/2}\,, \tag{4}$$

where $C > 0$ *denotes an absolute constant.*

Motivation

- Wiener classes (isotropic) and their embeddings have been studied by Temlayakov, Krieg and others
- Nguyen Nguyen and Sickel recently studied some s-numbers of mixed Wiener classes in [7], however they studied neither Gelfand numbers, sampling numbers nor best m-term approximation
- new results concerning sampling numbers

Proposition 2 ([4, Jahn, T. Ullrich and Voigtlaender 2023])

Let $n, d \in \mathbb{N}$ then it holds for a quasi-normed function space with continuous embedding into L_∞

$$\varrho_{nd \log(d) \log(n)^2 \log(N)} \left(\mathcal{F}\right)_2 \leq C \sigma_n \left(\mathcal{F}\right)_\infty + E_{[-N,N]^d}(\mathcal{F})_\infty. \tag{5}$$

See also [5] by Krieg for a version with $\mathcal{A}$ as target space om the right hand side.

Relations between s-numbers

- Gelfand numbers form a lower bound for the non-linear sampling numbers, in particular it holds
$$\varrho_n\big(\mathcal{A}_p^\alpha(\mathbb{T}^d)\big)_2 \gtrsim c_n(\mathcal{A}_p^\alpha(\mathbb{T}^d))_2$$
- Kolmogorov numbers form a lower bound for the linear sampling numbers, in particular it holds
$$\varrho_n^{\mathrm{lin}}\big(\mathcal{A}_p^\alpha(\mathbb{T}^d)\big)_2 \gtrsim d_n(\mathcal{A}_p^\alpha(\mathbb{T}^d))_2$$

In total the Gelfand and sampling numbers give upper and lower bounds for the non-linear sampling numbers.

Mixed Wiener spaces

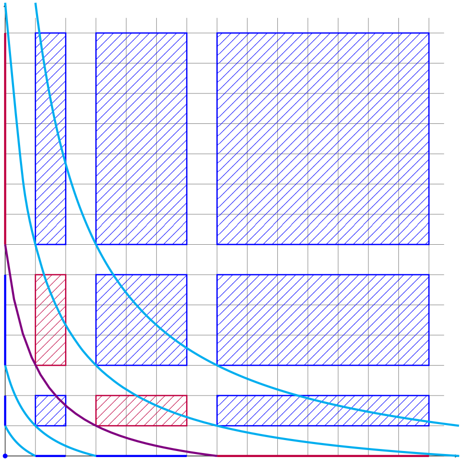
For $\alpha > 0$ and $0 < p < \infty$ we define the mixed Wiener space $\mathcal{A}_p^\alpha(\mathbb{T}^d) \subset L_1(\mathbb{T}^d)$ via its norm

$$\|f\|_{\mathcal{A}_p^\alpha(\mathbb{T}^d)} = \Big(\sum_{\mathbf{k}\in\mathbb{Z}^d} \prod_{i=1}^{d} (1+|k_i|)^{\alpha p} |\hat{f}(\mathbf{k})|^p \Big)^{\frac{1}{p}}.$$

For $p = 1$ these spaces are the periodic versions of Barron classes. The space $\mathcal{A}_1^0$ is the original Wiener Algebra $\mathcal{A}$. They have a useful embedding into the sequence spaces

$$A_\alpha f = \Big(\prod_{i=1}^{d} (1+|k_i|)^{\alpha} \hat{f}(\mathbf{k}) \Big)_{\mathbf{k}\in\mathbb{Z}^d}, \quad \|A_\alpha : \mathcal{A}_p^\alpha(\mathbb{T}^d) \to \ell_p(\mathbb{Z}^d)\| = 1.$$

Hyperbolic cross



A hyperbolic cross is a set of the form
$\left\{\mathbf{n} \in \mathbb{N}_0^d \middle| \prod_{j=1}^d (n_j + 1) \leq c\right\}$.
These are exactly the balls of this norm on the frequency side (for functions with only one Fourier coefficient).
For our proofs we are particularly interested in dyadic hyperbolic crosses.

Theorem 3

For $n, d \in \mathbb{N}$, $0 < p \leq 2$ and $\alpha > \left(\frac{p-1}{p}\right)_+$ it holds

$$c_n(\mathcal{A}_p^\alpha(\mathbb{T}^d))_2 \asymp n^{-(\alpha+\frac{1}{p}-\frac{1}{2})} \log(n)^{(d-1)\alpha}. \tag{6}$$

Theorem 4

For $n, d \in \mathbb{N}$ with $0 < p \leq q$ and $2 \leq q \leq \infty$ as well as $\alpha > \left(\frac{p-1}{p}\right)_+$ it holds

$$n^{-(\alpha+\frac{1}{p}-\frac{1}{2})} \log(n)^{(d-1)\alpha} \lesssim \sigma_n\big(\mathcal{A}_p^\alpha(\mathbb{T}^d)\big)_q \lesssim n^{-(\alpha+\frac{1}{p}-\frac{1}{2})} \log(n)^{(d-1)\alpha+\mu} \tag{7}$$

where $\mu = 1$ if both $q = \infty$ and $d > 1$ otherwise $\mu = 0$.

Linear sampling numbers

Proposition 5 (see [7, Nguyen, Nguyen and Sickel, 2022])

For the Kolmogorov numbers d_n it holds for $\alpha > 0$,

$$d_n\big(\mathcal{A}_1^\alpha(\mathbb{T}^d)\big)_2 \asymp n^{-\alpha} \log(n)^{\alpha(d-1)}. \tag{8}$$

Since the Kolmogorov numbers form a lower bound for the linear sampling numbers this immediately gives the following result

$$\varrho_n^{\mathrm{lin}}\big(\mathcal{A}_1^\alpha(\mathbb{T}^d)\big)_2 \gtrsim n^{-\alpha} \log(n)^{\alpha(d-1)}. \tag{9}$$

Non-linear sampling numbers

For the non-linear sampling numbers an analogous bound holds in terms of the Gelfand numbers

$$\varrho_n\big(\mathcal{A}_1^\alpha(\mathbb{T}^d)\big)_2 \gtrsim n^{-(\alpha+\frac{1}{2})}\log(n)^{\alpha(d-1)}. \tag{10}$$

Proposition 2 together with Theorem 4 now yields

$$\varrho_n\big(\mathcal{A}_1^\alpha(\mathbb{T}^d)\big)_2 \lesssim n^{-(\alpha+\frac{1}{2})}\log(n)^{\alpha(d-1)+3(\alpha+\frac{1}{2})+\frac{1}{2}} \tag{11}$$

There is a difference of $\frac{1}{2}$ in the main rate of the decay between the linear and non-linear sampling numbers in mixed Wiener classes measured in L_2.

Best m-term approximation of $\mathcal{A}$

Lemma 6

Let $2 \leq q < \infty$ and $\alpha > 0$ then it holds

$$\sigma_n(\mathcal{A})_q \leq C \frac{q}{\log(q)} n^{-\frac{1}{2}} \tag{12}$$

for an absolute constant $C \geq 1$.

We can even employ the Nikolskij inequality to get a version of this for $q = \infty$.

Lemma 7

For $N \in \mathbb{N}$ and a trigonometric polynomial $t \in \mathcal{T}([-N, N]^d)$ it holds

$$\sigma_n(t)_\infty \leq C d \log(N) n^{-\frac{1}{2}} \|t\|_{\mathcal{A}}. \tag{13}$$

Tractable bound on the best m-term approximation

Again the original Theorem 4 states

$$\sigma_n\big(\mathcal{A}_1^\alpha(\mathbb{T}^d)\big)_\infty \lesssim n^{-(\alpha+\frac{1}{2})}\log(n)^{(d-1)\alpha+1}$$

Where another 2^d term is hidden by the $\lesssim$. This is not a suitable bound in a setting where $n = d^s$.

Theorem 8

Let $m, d \in \mathbb{N}$ *and* $\alpha > 0$ *then it holds*

$$\sigma_n\big(\mathcal{A}_1^\alpha(\mathbb{T}^d)\big)_\infty \le Cn^{-\frac{1}{2}}d\log(n)^{1/2} \tag{14}$$

with absolute constant $C \geq 1$.

By using [2]. This bound decays for $n > d^2$.

Besov spaces

For Besov spaces with dominating mixed smoothness

$$S^r_{p,\theta}B(\mathbb{T}^d) := \Big\{ f \in L_p(\mathbb{T}^d) \;:\; \Big(\sum_{\mathbf{l}\in\mathbb{N}_0^d} 2^{|\mathbf{l}|_1 r\theta} \Big\| \sum_{\mathbf{k}\in I_{\mathbf{l}}} \hat{f}(\mathbf{k}) \exp(2\pi i \mathbf{k}\mathbf{x}) \Big\|_p^\theta \Big)^{\frac{1}{\theta}} < \infty \Big\}, \tag{15}$$

we get with $I_0 = \{0\}^d$ and for $\mathbf{l} \in \mathbb{N}^d$,

$$I_{\mathbf{l}} = \big\{ k \in \mathbb{Z} \;:\; 2^{l_1-1} \le |k| < 2^{l_1} \big\} \times \cdots \times \big\{ k \in \mathbb{Z} \;:\; 2^{l_d-1} \le |k| < 2^{l_d} \big\}.$$

Theorem 9

Let $n, d \in \mathbb{N}$ and $(p,\theta) \in \{(p,\theta) : 2 \le p < \infty, 0 < \theta \le 1\} \setminus (2,1)$. Then it holds

$$\sigma_n(S^{1/\theta-1/2}_{p,\theta}B(\mathbb{T}^d))_\infty \le C\,(1/\theta - 1/2 - 1/p)^{-1/2} d\, n^{1/2-1/\theta} \log(dn)^{1/2}, \tag{16}$$

where $C > 0$ denotes an absolute constant.

Besov spaces embedded into Wiener spaces

Idea of proof: The space $S^{1/\theta-1/2}_{p,\theta}B(\mathbb{T}^d)$ is embedded into $S^0_\theta\mathcal{A}(\mathbb{T}^d)$ for $2 \leq p < \infty$ and $0 < \theta \leq 1$ with constant operator norm.

$$\begin{array}{ccc} S^{1/\theta-1/2}_{p,\theta}B & \xrightarrow{\quad ? \quad} & L_\infty \\ {\scriptstyle 1}\Big\downarrow & & \Big\uparrow{\scriptstyle m^{-1/2}} \\ S^0_\theta\mathcal{A} & \xrightarrow{\; m^{1-1/\theta} \;} & S^0_1\mathcal{A} \end{array}$$

Where the embedding $S^0_\theta\mathcal{A} \to S^0_1\mathcal{A}$ is simply the Stechkin Lemma* [9].

Thank you for your attention

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